

Approximating the Geometry of Dynamics in Potential Games

Point-wise Convergence, Regions of Attraction, Average Case Performance Analysis,
and System Invariants

Ioannis Panageas
Georgia Institute of Technology
ioannis@gatech.edu

Georgios Piliouras
Singapore University of Technology and Design
georgios.piliouras@gmail.com

Abstract

What does it mean to fully understand the behavior of a network of competing agents? The golden standard typically is the behavior of learning dynamics in potential games, where many evolutionary dynamics, e.g., replicator, are known to converge to sets of equilibria. Even in such classic settings many critical questions remain unanswered. Inspired by topological and geometric considerations, we devise novel yardsticks and techniques that allow for much more detailed analysis of game dynamics and network computation more generally. We address issues such as

- *Point-wise convergence:* Does the system actually equilibrate even in the presence of continuums of equilibria?
- *Average case analysis:* How does the system behave given a uniformly random initial condition (or more generally an initial condition chosen from a given distribution)? How does this “average case” performance compare against standard measures of efficiency such as worst case (price of anarchy) and best case analysis (price of stability)?
- *Computing regions of attraction:* Given point-wise convergence can we compute the region of asymptotic stability of each equilibrium (e.g., estimate its volume, geometry)?
- *System invariants:* A system invariant is a function defined over the system state space such that it remains constant along every system trajectory. In some sense, this is a notion that is orthogonal to the classic game theoretic concept of a potential function, which always strictly increases/decreases along system trajectories. Do dynamics in a potential game exhibit (besides the standard potential function) non-trivial invariant functions as well? If so, how many? How do these functions look like?

All the above issues prove to be tightly coupled to each other and open possibilities for a deeper understanding of many settings of networked, decentralized computation.

1 Introduction

The study of game dynamics is a basic staple of game theory. The history of the field traces back to the early work of Brown and Robinson [6, 30] on learning dynamics in zero-sum games, which itself came shortly on the heels of von Neumann’s foundational investigation of equilibrium computation. Since then vast amount of work has accumulated on the subject with several books dedicated to it [39, 34, 15, 38].

A key question when it comes to the study of any system is what constitutes a sufficient understanding of its behavior. In the case of learning in games the golden standard is to establish convergence to equilibria. Thus, it is hardly surprising that a significant part of the work on learning in games focuses on potential games (and slight generalizations thereof) where many dynamics (e.g. replicator, smooth fictitious play) are known to converge to equilibrium sets. The structure of the convergence proofs is essentially universal across different learning dynamics and boils down to identifying a Lyapunov/potential function that strictly decreases along any nontrivial trajectory. In potential games, as their name suggests, this function is part of the description of the game and precisely guides self-interested dynamics towards critical points of these functions that correspond to equilibria of the learning process.

Potential games are also isomorphic to congestion games [24]. Congestion games have been instrumental in the study of efficiency issues in games. They are amongst the most extensively studied class of games from the perspective of price of anarchy and price of stability with many tight characterization results for different subclasses of games (e.g., linear congestion games [33], symmetric load balancing [26] and references therein).

Given this extensive treatment of such a classic class of games it would seem, at a first glance, that our understanding of these dynamical systems is more or less complete. Our goal in this paper is to show that this is far from true. We focus on simple systems where replicator dynamic is applied to linear congestion games and (network) coordination games. We establish a number of novel results along with new benchmarks and yardsticks for a much tighter understanding of learning in games. Our hope is that these approaches can be pursued in wider classes of learning in games and hopefully more generally in different settings of networked computation (such as in the study of biological evolution, see related work section).

Our results can be organized around four key ex: point-wise convergence to equilibrium, regions of attraction, average case performance analysis, and system invariants. We examine each of them next:

Trajectory characterizations beyond convergence to (sets of) equilibria: Point-wise convergence (to equilibrium). Our first goal is to prove system convergence to equilibrium instead of equilibrium sets. We start by elucidating the difference between these two statements. Although they are virtually identical from a linguistic perspective, they are quite distinct from a formal, topological perspective. Convergence to equilibrium sets implies that the distance of system trajectories from the sets of equilibria converges to zero. Convergence to equilibrium, also referred as to point-wise convergence, implies that every system trajectory has a unique limit point, which is an equilibrium. In games with continuum of equilibria, (e.g., n balls n bins games with $n \geq 4$), the first statement is more inclusive than the second. In fact, system equilibration is not implied by set-wise convergence, and the limit set of a trajectory may have complex topology (e.g., the limit of social welfare may not be well defined).

In order to get an intuitive understanding of these issues, let’s examine a variant of Zeno’s paradox of the tortoise and Achilles. In a twist of the typical narrative, now at the end of the t -phase of the run, instead of the tortoise gaining a $1/2^t$ advantage over Achilles, our faster tortoise has a $1/t$ length advantage over Achilles. In this case, the series diverges and Achilles actually does not reach the tortoise (i.e., his trajectory up to reaching the tortoise is infinitely long). Similarly, we may have trajectories that converge to an equilibrium set, however the speed with which they move converges to zero slowly. It is still possible for such trajectories to

have infinite length and span complex patterns in the vicinity of a continuum of an equilibrium set. The system behavior may be rather complex, despite the seemingly reassuring convergence to equilibrium sets statement. In fact, such a statement is borderline inaccurate since it invites the false presumption that the system actually converges, when in fact it may well not!

We establish point-wise convergence for replicator dynamics in general linear congestion games and arbitrary networks of coordination games. Despite numerous positive convergence results in classes of congestion games ([13, 4, 12, 3, 1]), this is the first to our knowledge result about deterministic point-wise convergence for any concurrent dynamic. The proof is based on local, information theoretic Lyapunov functions around each equilibrium and not on the typical global potential functions used in establishing the standard set-wise convergence to equilibria.

Behavioral predictions beyond equilibrium selection: Computing regions of attraction. There is a long history of applying learning dynamics as selection mechanisms in games with multiple equilibria [39]. The first step along those lines is to distinguish between equilibria whose region of attraction have positive/zero measure. In potential/congestion games, all equilibria with no unstable eigenvectors must satisfy a necessary game theoretic condition, known as weak stability [18]. A Nash equilibrium is weakly stable if given any two randomizing agents, fixing one of the agents to choosing one of his strategies with probability one, leaves the other agent indifferent between the strategies in his support. *Given point-wise convergence of the replicator, all but a zero measurable subset of initial conditions converge to weakly stable equilibria.* However, this is once again merely a point of departure for our investigation, since we do not merely to partition equilibria into admissible and non-admissible but actually quantify their likelihood as it is captured by the size of their regions of attraction.

Risk dominance is an equilibrium refinement process that centers around uncertainty about opponent behavior. A Nash equilibrium is considered risk dominant if it has the largest basin of attraction¹. The benchmark example is the Stag Hunt game, shown in figure 1(a). In such symmetric 2x2 coordination games a strategy is risk dominant if it is a best response to the uniformly random strategy of the opponent. We show that *the likelihood of the risk dominant equilibrium of the Stag Hunt game* is $\frac{1}{27}(9 + 2\sqrt{3}\pi) \approx 0.7364$ (instead of merely knowing that it is at least 1/2, see figure 2). The size of the region of attraction of the risk dominated equilibrium is 0.2636, whereas the mixed equilibrium has region of attraction of zero measure. Moving to networks of coordination games, we show how to *construct an oracle that predicts the limit behavior of an arbitrary initial condition*, in the case of coordination games played over a star network with n agents. This is the most economic class of games that exhibits two analytical obstacles that intuitively are thought to pose intractable obstacles in the quantitative analysis of nonlinear systems: i) they have n (polynomially many) free variables ii) they exhibit a continuum of equilibria.

Efficiency analysis beyond price of anarchy, stability: Average case performance. A related challenge in the study of any system of learning dynamics is to faithfully capture its efficiency. Given the stationarity of equilibria as well as our convergence results, both worst case as well as best case system performance are captured by the price of anarchy and price of stability respectively. It is clear, however, that both extremes, best and worst case may not be good representatives of average case system performance. If we assume that the initial conditions are chosen uniformly at random² then the expected system limit performance is captured by a weighted average where the social welfare of each equilibrium is weighted proportionally to its volume of its region of attraction.

We analyze the average case performance of a class of *two agent coordination games* that generalizes Stag Hunt games/dynamics. We compute upper and lower bounds for the volume of

¹Although risk dominance [14] was originally introduced as a hypothetical model of the method by which perfectly rational players select their actions, it may also be interpreted [25] as the result of evolutionary processes.

²The assumption of uniform initial conditions is for canonical purposes and all results can easily be extended to any choice of distribution.

the region of attraction of each equilibrium using polytope approximations. The *average case performance* of this class of games, where the social welfare of each equilibrium is weighted by the size of its basin *is within 1.15 and 1.21 of optimal*. In contrast, the *price of anarchy is unbounded*. We also study *n-balls, n bins games*. For $n \geq 4$, we show that such games have uncountably many (randomized) equilibria. It is well known that such symmetric games exhibit a fully symmetric Nash equilibrium where each ball/agent chooses a bin uniformly at random. Its makespan is $\Theta(\log(n)/\log \log(n))$ and is equal to its price of anarchy. The replication dynamic is shown to lead to asymmetric equilibria. In fact, the region of attraction of all randomized equilibria has zero measure. As a result, the *average system performance* both in terms of social cost as well as makespan is *equal to one*. This result is robust for any continuous distribution of initial conditions³. Finally, we show that the average system performance of any *linear congestion game* is strictly less than $5/2$ times the optimal.

System invariants: A system invariant is a function defined over the system state space such that it remains constant along every system trajectory. In some sense, this is a notion that is orthogonal to the classic game theoretic concept of a potential function, which always strictly increases/decreases along system trajectories. Do dynamics in a potential game exhibit (besides the standard potential function) non-trivial invariant functions as well? If so, how many? How do these functions look like?

We identify information theoretic invariant properties of replicator dynamics for several large scale systems. In the case of bipartite coordination games with fully mixed Nash equilibria, the difference between the sum of the Kullback-Leibler (K-L) divergences of the evolving mixed strategies of the agents on the left partition from their fully mixed Nash equilibrium strategy and the respective term for the agents in the right partition remains constant along any trajectory. In the special case of star graphs, we show how to produce n such invariants where n is the degree of the star. This allows for creating efficient oracles for predicting to which Nash equilibrium the system converges provably without simulating explicitly the system trajectory. Abstractly, this analysis can be thought of simulating efficiently the job of physicist in terms of identifying the maximal number of physical laws satisfied by a given system.

2 Related Work

Set-wise convergence in congestion/potential games: In [31] Rosenthal showed that every congestion game has pure Nash equilibria, and that better-response dynamics converge to them. In these dynamics, in every round, exactly one agent deviates to a better strategy. If two or more agents move at the same time then convergence is not guaranteed. Recently, a number of positive convergence results have been established for concurrent dynamics [13, 4, 12, 3, 1, 18], however, they usually depend on strong assumptions about network structure (*e.g.* load balancing games) and/or symmetry of available strategies and/or are probabilistic in nature and/or establish convergence to approximate equilibria. On the contrary our convergence results are deterministic, hold for any network structure and in the case of the replicator dynamic are *point-wise*.

Evolution in Computer Science: Replicator & Coordination Games. In the last couple of years several theoretical results have been proven on the intersection of computer science and evolution [20, 36, 37, 23, 22]. Many of these directly regard (variants of) replicator dynamics applied to coordination games. In [8, 7] Chastain *et al.* show that standard models of haploid evolution can be directly interpreted as (discrete) replicator dynamics [15] employed in coordination games. Analogous connections to replicator and coordination games are also known for models of diploid sexual evolution [21] along with point-wise convergence results. (Discrete) replicator dynamics are closely connected to the family of multiplicative weights

³These results also extend in a straightforward manner to games of n balls, m bins with $n \neq m$. The optimality in terms of makespan generically extends even if we allow different speeds on each machine.

update algorithm (MWUA) [18]. Our results could help shed more light in the study of evolution and genetics.

Learning as a refinement mechanism: Price of anarchy-like bounds in potential games using equilibrium stability refinements (*e.g.*, stochastically stable states) have been explored before [10, 2, 1]. Our approach and techniques are more expansive in scope, since they also allow for computing the actual likelihoods of each equilibrium as well as the topology of the regions of attractions of different equilibria.

This paper builds upon recent positive performance results for the replicator dynamics (and discrete-time variants). The key reference is [18], where many key ideas including the fact that replicator dynamics can significantly outperform worst case equilibria were introduced. Replicator can outperform even best case equilibria by converging to cycles [17, 19].

3 Concepts and Toolbox

3.1 Congestion/Network Coordination Games

3.1.1 Congestion Games

A *congestion game* [31] is defined by the tuple $(N; E; (S_i)_{i \in N}; (c_e)_{e \in E})$ where N is the set of *agents*, E is a set of *resources* (also known as *edges* or *bins* or *facilities*), and each player i has a set S_i of subsets of E ($S_i \subseteq 2^E$) and $|S_i| \geq 2$. Each strategy $s_i \in S_i$ is a set of edges (a *path*), and c_e is a cost (negative utility) function associated with facility e . We will also use small greek characters like γ, δ to denote different strategies/paths. For a strategy profile $s = (s_1, s_2, \dots, s_N)$, the cost of player i is given by $c_i(s) = \sum_{e \in s_i} c_e(\ell_e(s))$, where $\ell_e(s)$ is the number of players using e in s (the load of edge e). In linear congestion games, the latency functions are of the form $c_e(x) = a_e x + b_e$ where $a_e, b_e \geq 0$. Furthermore, the social cost will correspond to the sum of the costs of all the agents $sc = \sum_i c_i(s)$.

3.1.2 Network (Polymatrix) Coordination Games

An n -player polymatrix (network) coordination game is defined by an undirected graph $G(V, E)$ with $|V| = n$ vertices and each vertex corresponds to a player. An edge $(i, j) \in E(G)$ corresponds to a coordination game between players i, j . We assume that we have the same strategy space S for every edge. Let A_{ij} be the payoff matrix for the game between players i, j and $A_{ij}^{\gamma\delta}$ be the payoff for both (coordination) if i, j choose strategies γ, δ respectively. The set of players will be denoted by N and the set of neighbors of player i will be denoted by $N(i)$. For a strategy profile $s = (s_1, s_2, \dots, s_N)$, the utility of player i is given by $u_i(s) = \sum_{j \in N(i)} A_{ij}^{s_i s_j}$. The social welfare of a state corresponds to the sum of the utilities of all the agents $sw = \sum_i u_i$.

The price of anarchy is defined as: $\text{PoA} = \frac{\max_{\sigma \in \text{NE}} \text{Social Cost}(\sigma)}{\min_{\sigma^* \in \times_i S_i} \text{Social Cost}(\sigma^*)}$. for cost functions and similarly $\text{PoA} = \frac{\max_{\sigma^* \in \times_i S_i} \text{Social Welfare}(\sigma^*)}{\min_{\sigma \in \text{NE}} \text{Social Welfare}(\sigma)}$ for utilities.

3.2 Replicator Dynamics

The replicator equation is described by the following differential equation:

$$\frac{dp_i(t)}{dt} = \dot{p}_i = p_i[u_i(p) - \hat{u}(p)], \quad \hat{u}(p) = \sum_{i=1}^n p_i u_i(p)$$

where p_i is the proportion of type i in the population, $p = (p_1, \dots, p_m)$ is the vector of the distribution of types in the population, $u_i(p)$ is the fitness of type i , and $\hat{u}(p)$ is the average population fitness. The state vector p can also be interpreted as a randomized strategy. In

the case of costs, our model is captured by the following system: $\frac{dp_{i\gamma}}{dt} = p_{i\gamma}(\hat{c}_i - c_{i\gamma})$ for each $i \in N$, $\gamma \in S_i$, where $p_{i\gamma}$ is the probability player i chooses strategy γ , $c_{i\gamma} = \mathbb{E}_{s_{-i} \sim p_{-i}} c_i(\gamma, s_{-i})$ and $\hat{c}_i = \sum_{\delta} p_{i\delta} c_{i\delta}$. In the case of utilities, our model is captured by the following system: $\frac{dp_{i\gamma}}{dt} = p_{i\gamma}(u_{i\gamma} - \hat{u}_i)$ for each $i \in N$, $\gamma \in S_i$, where $p_{i\gamma}$ is the probability player i chooses strategy γ , $u_{i\gamma} = \mathbb{E}_{s_{-i} \sim p_{-i}} u_i(\gamma, s_{-i})$ and $\hat{u}_i = \sum_{\delta} p_{i\delta} u_{i\delta}$. We will denote $\Delta(S_i) = \{\mathbf{p} : \sum_{\gamma} p_{i\gamma} = 1\}$, $\Delta = \times_i \Delta(S_i)$.

Remark: A fixed point of a flow is a point where the vector field is equal to zero. An interesting observation about the replicator is that its fixed points are exactly the set of randomized strategies such that each agent experiences equal costs/utilities across all strategies he chooses with positive probability. This is a generalization of the notion of Nash equilibrium, since equilibria furthermore require that any strategy that is played with zero probability must have expected cost (utility) at least as high (low) as those strategies which are played with positive probability.

3.3 Topology of dynamical systems

Our treatment follows that of [38], the standard text in evolutionary game theory, which itself borrows material from the classic book by Bhatia and Szegő [5]. Since our state space is compact and the replicator vector field is Lipschitz-continuous, we can present the unique solution of our ordinary differential equation as a continuous map $\phi : \mathcal{S} \times \mathbb{R} \rightarrow \mathcal{S}$ called flow of the system. Fixing starting point $x \in \mathcal{S}$ defines a function of time which captures the trajectory (orbit, solution path) of the system with the given starting point. This corresponds to the graph of $\phi(x, \cdot) : \mathbb{R} \rightarrow \mathcal{S}$, i.e., the set $\{(t, y) : y = \phi(x, t) \text{ for some } t \in \mathbb{R}\}$.

If the starting point x does not correspond to an equilibrium then we wish to capture the asymptotic behavior of the system (informally the limit of $\phi(x, t)$ when t goes to infinity). Typically, however, such functions do not exhibit a unique limit point so instead we study the set of limits of all possible convergent subsequences. Formally, given a dynamical system $(\mathbb{R}, \mathcal{S}, \phi)$ with flow $\phi : \mathcal{S} \times \mathbb{R} \rightarrow \mathcal{S}$ and a starting point $x \in \mathcal{S}$, we call point $y \in \mathcal{S}$ an ω -limit point (or merely limit point) of the orbit through x if there exists a sequence $(t_n)_{n \in \mathbb{N}} \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} t_n = \infty$, $\lim_{n \rightarrow \infty} \phi(x, t_n) = y$. Alternatively the ω -limit set can be defined as: $\omega_\phi(x) = \bigcap_t \bigcup_{\tau \geq t} \phi(x, \tau)$.

A Lyapunov (or potential) function is a function that strictly decreases along every non-trivial trajectory of the dynamical system. Lyapunov functions are typically used to establish convergence and stability results in learning in games and more generally in dynamical systems. Moreover, informally the stable manifold of an equilibrium corresponds to the set of states which converge to it as $t \rightarrow \infty$, whereas the unstable manifold to those states which converge to it in the reverse direction of the flow, i.e., as $t \rightarrow -\infty$ (see [38, 5] for more details). Finally, throughout the paper when we talk about a stable fixed point \mathbf{q} we mean that the Jacobian of the rule of the dynamics at \mathbf{q} has eigenvalues with real part (for continuous systems), otherwise we call it unstable.

4 Point-wise convergence and Stability

In this section, we prove point-wise convergence for replicator dynamics in linear congestion games as well as in networks of coordination games. Furthermore, we establish via stability analysis that for all but a zero-measure of initial conditions these system converge (point-wise) to restricted classes of equilibria.

Theorem 1. *For any linear congestion game, replicator dynamics converges to a fixed point (point-wise convergence).*

The proof has two steps. The first step which is already known (see [18]) is to construct a Lyapunov function, i.e., a function that is decreasing along the trajectories of the dynamics. The Lyapunov function will be a potential function for the congestion game. This shows that the dynamics converge to equilibria sets. The second step is to construct a *local* Lyapunov function in some small neighborhood of a limit point. Formally:

Proof We observe that $\Psi(\mathbf{p}) = \sum_i \hat{c}_i + \sum_{i,\gamma} \sum_{e \in \gamma} (b_e + a_e) p_{i\gamma}$ is a Lyapunov function since

$$\frac{\partial \Psi}{\partial p_{i\gamma}} = c_{i\gamma} + \sum_{j \neq i} p_{j\gamma'} \frac{\partial c_{j\gamma'}}{\partial p_{i\gamma}} + \sum_{e \in \gamma} (b_e + a_e) = c_{i\gamma} + \underbrace{\sum_{j \neq i} \sum_{\gamma'} \sum_{e \in \gamma \cap \gamma'} a_e p_{j\gamma'}}_{c_{i\gamma}} + \sum_{e \in \gamma} (b_e + a_e) = 2c_{i\gamma}$$

and hence $\frac{d\Psi}{dt} = \sum_{i,\gamma} \frac{\partial \Psi}{\partial p_{i\gamma}} \frac{dp_{i\gamma}}{dt} = - \sum_{i,\gamma,\gamma'} p_{i\gamma} p_{i\gamma'} (c_{i\gamma} - c_{i\gamma'})^2 \leq 0$, with equality at fixed points. Hence (as in [18]) we have convergence to equilibria sets (compact connected sets consisting of fixed points). This doesn't suffice for point-wise convergence. To be exact it suffices only in the case the equilibria are discrete (which is not the case for linear congestion games - see lemma 16).

Let \mathbf{q} be a limit point of the trajectory $\mathbf{p}(t)$ where $\mathbf{p}(t)$ is in the interior of Δ for all $t \in \mathbb{R}$ (since we started from an initial condition inside Δ) then we have that $\Psi(\mathbf{q}) < \Psi(\mathbf{p}(t))$. We define the relative entropy $I(\mathbf{p}) = - \sum_i \sum_{\gamma: q_{i\gamma} > 0} q_{i\gamma} \ln(p_{i\gamma}/q_{i\gamma}) \geq 0$ (Jensen's ineq.) and $I(\mathbf{p}) = 0$ iff $\mathbf{p} = \mathbf{q}$. We get that

$$\begin{aligned} \frac{dI}{dt} &= - \sum_i \sum_{\gamma: q_{i\gamma} > 0} q_{i\gamma} (\hat{c}_i - c_{i\gamma}) = - \sum_i \hat{c}_i + \sum_{i,\gamma} q_{i\gamma} c_{i\gamma} \\ &= - \sum_i \hat{c}_i + \sum_{i,\gamma} \sum_{e \in \gamma} (b_e + a_e) q_{i\gamma} + \sum_{i,\gamma} \sum_{j \neq i} \sum_{\gamma'} \sum_{e \in \gamma \cap \gamma'} a_e q_{i\gamma} p_{j\gamma'} \\ &= - \sum_i \hat{c}_i + \sum_{i,\gamma} \sum_{e \in \gamma} (b_e + a_e) q_{i\gamma} + \sum_{i,\gamma} \sum_{j \neq i} \sum_{\gamma'} \sum_{e \in \gamma \cap \gamma'} a_e q_{j\gamma'} p_{i\gamma} \\ &= - \sum_i \hat{c}_i + \sum_{i,\gamma} \sum_{e \in \gamma} (b_e + a_e) q_{i\gamma} - \sum_{i,\gamma} \sum_{e \in \gamma} (b_e + a_e) p_{i\gamma} + \sum_{i,\gamma} p_{i\gamma} (d_{i\gamma}) \\ &= \sum_i \hat{d}_i - \sum_i \hat{c}_i + \sum_{i,\gamma} \sum_{e \in \gamma} (b_e + a_e) q_{i\gamma} - \sum_{i,\gamma} \sum_{e \in \gamma} (b_e + a_e) p_{i\gamma} - \sum_{i,\gamma} p_{i\gamma} (\hat{d}_i - d_{i\gamma}) \\ &= \Psi(\mathbf{q}) - \Psi(\mathbf{p}) - \sum_{i,\gamma} p_{i\gamma} (\hat{d}_i - d_{i\gamma}) \end{aligned}$$

where $d_{i\gamma}, \hat{d}_i$ correspond to the cost of player i if he chooses strategy γ and his expected cost respectively at point \mathbf{q} . The rest follows similarly to [21].

We break the term $\sum_{i,\gamma} p_{i\gamma} (\hat{d}_i - d_{i\gamma})$ to positive and negative terms (the zero terms can be ignored), i.e., $\sum_{i,\gamma} p_{i\gamma} (\hat{d}_i - d_{i\gamma}) = \sum_{i,\gamma: \hat{d}_i > d_{i\gamma}} p_{i\gamma} (\hat{d}_i - d_{i\gamma}) + \sum_{i,\gamma: \hat{d}_i < d_{i\gamma}} p_{i\gamma} (\hat{d}_i - d_{i\gamma})$

Claim: There exists an $\epsilon > 0$ so that the function $Z(\mathbf{p}) = I(\mathbf{p}) + 2 \sum_{i,\gamma: \hat{d}_i < d_{i\gamma}} p_{i\gamma}$ has $\frac{dZ}{dt} < 0$ for $|\mathbf{p} - \mathbf{q}| < \epsilon$ and $\Psi(\mathbf{q}) < \Psi(\mathbf{p})$.

Assuming that $\mathbf{p} \rightarrow \mathbf{q}$, we get $\hat{c}_i - c_{i\gamma} \rightarrow \hat{d}_i - d_{i\gamma}$ for all i, γ . Hence for small enough $\epsilon > 0$ with

$|\mathbf{p} - \mathbf{q}| < \epsilon$, we have that $\hat{c}_i - c_{i\gamma} \leq \frac{3}{4}(\hat{d}_i - d_{i\gamma})$ for the terms which $\hat{d}_i - d_{i\gamma} < 0$. Therefore

$$\begin{aligned}
\frac{dZ}{dt} &= \Psi(\mathbf{q}) - \Psi(\mathbf{p}) - \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) - \sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) + 2 \sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(\hat{c}_i - c_{i\gamma}) \\
&\leq \Psi(\mathbf{q}) - \Psi(\mathbf{p}) - \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) - \sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) + 3/2 \sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma}) \\
&= \underbrace{\Psi(\mathbf{q}) - \Psi(\mathbf{p})}_{<0} + \underbrace{\sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} -p_{i\gamma}(\hat{d}_i - d_{i\gamma})}_{\leq 0} + 1/2 \underbrace{\sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(\hat{d}_i - d_{i\gamma})}_{\leq 0} < 0
\end{aligned}$$

where we substitute $\frac{p_{i\gamma}}{dt} = p_{i\gamma}(\hat{c}_i - c_{i\gamma})$ (replicator), and the claim is proved.

Notice that $Z(\mathbf{p}) \geq 0^4$ and is zero iff $\mathbf{p} = \mathbf{q}$. (i)

To finish the proof of the theorem, if \mathbf{q} is a limit point of $\mathbf{p}(t)$, there exists an increasing sequence of times t_i , with $t_n \rightarrow \infty$ and $\mathbf{p}(t_n) \rightarrow \mathbf{q}$. We consider ϵ' such that the set $C = \{\mathbf{p} : Z(\mathbf{p}) < \epsilon'\}$ is inside $B = \{\mathbf{p} : |\mathbf{p} - \mathbf{q}| < \epsilon\}$ where ϵ is from claim above. Since $\mathbf{p}(t_n) \rightarrow \mathbf{q}$, consider a time t_N where $\mathbf{p}(t_N)$ is inside C . From the claim above we get that $Z(\mathbf{p})$ is decreasing inside B (and hence inside C), thus $Z(\mathbf{p}(t)) \leq Z(\mathbf{p}(t_N)) < \epsilon'$ for all $t \geq t_N$, hence the orbit will remain in C . By the fact that $Z(\mathbf{p}(t))$ is decreasing in C (claim above) and also $Z(\mathbf{p}(t_n)) \rightarrow Z(\mathbf{q}) = 0$ it follows that $Z(\mathbf{p}(t)) \rightarrow 0$ as $t \rightarrow \infty$. Hence $\mathbf{p}(t) \rightarrow \mathbf{q}$ as $t \rightarrow \infty$ using (i). ■

The same result can be shown for network coordination games. Formally:

Theorem 2. *For any network coordination game, replicator dynamics converges to a fixed point (point-wise convergence).*

The proof is almost the same as the proof of theorem 1 and can be found in the appendix. Replicator dynamics - in linear congestion games and network coordination games and essentially any dynamics that converge point-wise - induces a probability distribution over the fixed points. The probability for each fixed point is proportional to the volume of its region of attraction. Having this distribution over the fixed points, we can define other measurements like average performance. The fixed points can be exponentially many or even countable many, but as it is stated below (corollary 6), only the weakly stable Nash equilibria have non-zero volumes of attraction.

Definition 3. [18] *A Nash equilibrium is called weakly stable if given any two randomizing agents, fixing one of the agents to choosing one of his strategies with probability one, leaves the other agent indifferent between the strategies in his support.*

In [18] Kleinberg *et al.* showed that in congestion games, every stable fixed point is a weakly stable Nash equilibrium. The following theorem (that assumes point-wise convergence) has a corollary that for all but measure zero initial conditions, replicator dynamics converges to a weakly stable Nash equilibrium.

Theorem 4. *The set of initial conditions that converge to unstable fixed points has measure zero in Δ for linear congestion games and network coordination games.*

This theorem extends to all congestion games for which the replicator dynamics converges point-wise (e.g., systems with finite equilibria).

Theorem 5. [18] *In replicator dynamics on congestion games and network coordination games, every stable fixed point is a weakly stable Nash equilibrium.*

⁴sum of positive terms and $I(\mathbf{p}) \geq 0$

Proof In [18], the statement is proved for congestion games. Since a network coordination game is a potential game and every potential game is isomorphic to a congestion game, the rest follows. ■

Combining theorem 4 with theorem 5 we have the following corollary:

Corollary 6. *For all but measure zero initial conditions, replicator dynamics converges to weakly stable Nash equilibria for linear congestion games and network coordination games.*

Finally, before the end of this subsection, we present a lemma that gives a combinatorial characterization of the stable fixed points in network coordination games. First we have the following assumption (which is generic, eg considering random matrices)

Assumption 7. *For every $(i, j) \in E(G)$, each column and row of A_{ij} has distinct entries.*

Under this assumption it can be shown the following:

Lemma 8. *In the network coordination game $G(V, E)$ under the assumption 7 on the entries, for every weakly stable Nash equilibrium the set of randomized players forms an independent set in G .*

5 Average case performance/APoA

5.1 Average PoA in point-wise convergent systems

Let $\phi(t, \mathbf{p})$ which we also denote $\phi_t(\mathbf{p})$ be the flow of a differential equation $\dot{x} = f(x)$ where $f \in C^1(\Delta, \Delta)$. Under the assumption that the system converges point-wise, we said that the system induces a probability distribution over fixed points. Therefore, having a function social cost/welfare, we can quantify the behavior of the system, e.g compute the average social cost/welfare (under that distribution). Since a $\Delta_k = \{\sum_{i=1}^k x_i = 1, x_i \geq 0\}$ simplex has $k - 1$ dimensions, we denote $g(\Delta_k)$ the projection of domain Δ_k to R^{k-1} (exclude the first coordinate from k -tuple). If Δ is a product of simplices, $g(\Delta)$ is defined accordingly (e.g exclude the first coordinate from each player's corresponding tuple). We have the assumption that the social cost/welfare is continuous and defined on $g(\Delta)$.

Let $\psi(\mathbf{x}) = g(\lim_{t \rightarrow \infty} \phi_t(g^{-1}(\mathbf{x})))$, i.e., ψ returns the limit point (always exists by assumption) of $\mathbf{p} = g^{-1}(\mathbf{x})$ in $g(\Delta)$. We prove the following important lemma:

Lemma 9. *$\psi(\mathbf{x})$ is Lebesgue measurable.*

In case we have a discrete dynamical system $x(n+1) = f(x(n))$ for $n \in \mathbb{N}$ where f is C^1 and the system converges point-wise then $\psi(x) = \lim_n \underbrace{f \circ f \dots \circ f}_n(x)$ is measurable (same proof as in 9). Let f be a continuous or discrete probability density function on $g(\Delta)$. We define a new performance measure that we call f-PoA. In case f is the uniform distribution then we call it Average PoA. Essentially it compares the weighted average of all equilibrium points over the optimal solution. The respective weight of each equilibrium is just the measure of the points (over f) that converge to it. Formally:⁵

Definition 10. (*f-PoA*) *The f- PoA is the following Lebesgue integral (sc is continuous)*

$$\frac{\int_{g(\Delta)} sc \circ \psi(x) \cdot f(x) dx}{\min \text{ Social Cost}}$$

The integral above is well-defined since $\psi(\mathbf{x})$ is a Lebesgue measurable function and sc is a continuous function hence $sc \circ \psi$ is Lebesgue measurable.

⁵In the case of games that players want to maximize their utilities, we have the inverse fraction so that Average PoA ≥ 1 .

	Stag	Hare
Stag	5, 5	0, 4
Hare	4, 0	2, 2

(a) Stag hunt game.

	Stag	Hare
Stag	1, 1	0, 0
Hare	0, 0	w, w

(b) $G(w)$ coordination game

Figure 1

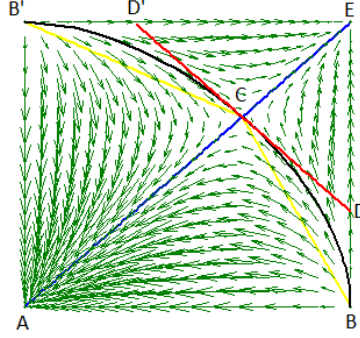


Figure 2: Vector field of replicator dynamics in Stag Hunt

In case f is the uniform distribution, the numerator is called the average performance and is essentially the expected value of $sc(X)$ where X follows the distribution that the dynamical system induces (probability distribution over fixed points, each fixed point has probability proportional to the measure of its region of attraction). Finally, observe that if f puts all the weight in the best/worse Nash equilibrium, the price of stability/anarchy is captured.

5.2 Regions of Attraction

5.2.1 Exact Quantitative Analysis of Risk Dominance in the Stag Hunt Game

The Stag Hunt game (figures 1(a)) has two pure Nash, $(Stag, Stag)$ and $(Hare, Hare)$ and a symmetric mixed Nash equilibrium with each agent choosing strategy *Hare* with probability $2/3$. Stag Hunt replicator trajectories are equivalent those of a coordination game⁶. Coordination games are potential games where the potential function in each state is equal to the utility of each agent. Replicator dynamic converges to pure Nash equilibria in this setting with probability 1. When we study the replicator dynamic here, it suffices to examine its projection in the subspace $p_{1s} \times p_{2s} \subset (0, 1)^2$ which captures the evolution of the probability that each agent assigns to strategy *Stag* (see figure 4). Using an invariant property of the dynamics, we compute the size of each region of attraction in this space and thus provide a quantitative analysis of risk dominance in the classic Stag Hunt game.

Theorem 11. *The region of attraction of $(Hare, Hare)$ is the subset of $(0, 1)^2$ that satisfies $p_{2s} < \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})$ and has Lebesgue measure $\frac{1}{27}(9 + 2\sqrt{3}\pi) \approx 0.7364$. The region of attraction of $(Stag, Stag)$ is the subset of $(0, 1)^2$ that satisfies $p_{2s} > \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})$ and has Lebesgue measure $\frac{1}{27}(18 - 2\sqrt{3}\pi) \approx 0.2636$. The stable manifold of the mixed Nash equilibrium satisfies the equation $p_{2s} = \frac{1}{2}(1 - p_{1s} + \sqrt{1 + 2p_{1s} - 3p_{1s}^2})$ and has 0 Lebesgue measure.*

⁶If each agent reduces their payoff in their first column by 4, these results to the depicted coordination game with $w = 2$.

Proof In the case of stag hunt games, one can verify in a straightforward manner (via substitution) that $\frac{d\left(\frac{2}{3}\ln(\phi_{1s}(\mathbf{p},t)) + \frac{1}{3}\ln(\phi_{1h}(\mathbf{p},t)) - \frac{2}{3}\ln(\phi_{2s}(\mathbf{p},t)) - \frac{1}{3}\ln(\phi_{2h}(\mathbf{p},t))\right)}{dt} = 0$, where $\phi_{i\gamma}(p, t)$, corresponds to the probability that each agent i assigns to strategy γ at time t given initial condition \mathbf{p} . We use this invariant to identify the stable and unstable manifold of the interior Nash \mathbf{q} .

Given any point \mathbf{p} of the stable manifold of \mathbf{q} , we have that by definition $\lim_{t \rightarrow \infty} \phi(\mathbf{p}, t) = \mathbf{q}$. Similarly for the unstable manifold, we have that $\lim_{t \rightarrow -\infty} \phi(\mathbf{p}, t) = \mathbf{q}$. The time-invariant property implies that for all such points (belonging to the stable or unstable manifold), $\frac{2}{3}\ln(p_{1s}) + \frac{1}{3}\ln(1-p_{1s}) - \frac{2}{3}\ln(p_{2s}) - \frac{1}{3}\ln(1-p_{2s}) = \frac{2}{3}\ln(q_{1h}) + \frac{1}{3}\ln(1-q_{1h}) - \frac{2}{3}\ln(q_{2h}) - \frac{1}{3}\ln(1-q_{2h}) = 0$, since the fully mixed Nash equilibrium is symmetric. This condition is equivalent to $p_{1s}^2(1-p_{1s}) = p_{2s}^2(1-p_{2s})$, where $0 < p_{1s}, p_{2s} < 1$. It is straightforward to verify that this algebraic equation is satisfied by the following two distinct solutions, the diagonal line ($p_{2s} = p_{1s}$) and $p_{2s} = \frac{1}{2}(1-p_{1s} + \sqrt{1+2p_{1s}-3p_{1s}^2})$. Below, we show that these manifolds correspond indeed to the state and unstable manifold of the mixed Nash equilibrium, by showing that the Nash equilibrium satisfies those equations and by establishing that the vector field is tangent everywhere along them.

The case of the diagonal is trivial and follows from the symmetric nature of the game. We verify the claims about $p_{2s} = \frac{1}{2}(1-p_{1s} + \sqrt{1+2p_{1s}-3p_{1s}^2})$. Indeed, the mixed equilibrium point in which $p_{1s} = p_{2s} = 2/3$ satisfies the above equation. We establish that the vector field is tangent to this manifold by showing in lemma 12 that $\frac{\partial p_{2s}}{\partial p_{1s}} = \frac{\zeta_{2s}}{\zeta_{1s}} = \frac{p_{2s}(u_2(s) - (p_{2s}u_2(s) + (1-p_{2s})u_2(h)))}{p_{1s}(u_1(s) - (p_{1s}u_1(s) + (1-p_{1s})u_1(h)))}$. Finally, this manifold is indeed attracting to the equilibrium. Since the function $p_{2s} = y(p_{1s}) = \frac{1}{2}(1-p_{1s} + \sqrt{1+2p_{1s}-3p_{1s}^2})$ is a strictly decreasing function of p_{1s} in $[0,1]$ and satisfies $y(2/3) = 2/3$, this implies that its graph is contained in the subspace $(0 < p_{1s} < 2/3 \cap 2/3 < p_{2s} < 1) \cup (2/3 < p_{1s} < 1 \cap 0 < p_{2s} < 2/3)$. In each of these subsets $(0 < p_{1s} < 2/3 \cap 2/3 < p_{2s} < 1)$, $(2/3 < p_{1s} < 1 \cap 0 < p_{2s} < 2/3)$ the replicator vector field coordinates have fixed signs that “push” p_{1s}, p_{2s} towards their respective equilibrium values.

The unstable manifold partitions the set $0 < p_{1s}, p_{2s} < 1$ into two subsets, each of which is flow invariant since the unstable manifold itself is flow invariant. Our convergence analysis for the generalized replicator flow implies that in each subset all but a measure zero of initial conditions must converge to its respective pure equilibrium. The size of the lower region of attraction⁷ is equal to the following definite integral $\int_0^1 \frac{1}{2}(1-p_{1s} + \sqrt{1+2p_{1s}-3p_{1s}^2})dx = \left[1/2\left(p_{1s} - \frac{p_{1s}^2}{2} + \left(-\frac{1}{6} + \frac{p_{1s}}{2}\right)\sqrt{1+2p_{1s}-3p_{1s}^2} - \frac{2\arcsin[\frac{1}{2}(1-3p_{1s})]}{3\sqrt{3}}\right)\right]_0^1 = \frac{1}{27}(9 + 2\sqrt{3}\pi) = 0.7364$ and the theorem follows. ■

We conclude by providing the proof of the following technical lemma:

Lemma 12. For any $0 < p_{1s}, p_{2s} < 1$, with $p_{2s} = \frac{1}{2}(1-p_{1s} + \sqrt{1+2p_{1s}-3p_{1s}^2})$ we have that:

$$\frac{\partial p_{2s}}{\partial p_{1s}} = \frac{\zeta_{2s}}{\zeta_{1s}} = \frac{p_{2s}(u_2(s) - (p_{2s}u_2(s) + (1-p_{2s})u_2(h)))}{p_{1s}(u_1(s) - (p_{1s}u_1(s) + (1-p_{1s})u_1(h)))}$$

Proof By substitution of the stag hunt game utilities, we have that:

$$\frac{\zeta_{2s}}{\zeta_{1s}} = \frac{p_{2s}(u_2(s) - (p_{2s}u_2(s) + (1-p_{2s})u_2(h)))}{p_{1s}(u_1(s) - (p_{1s}u_1(s) + (1-p_{1s})u_1(h)))} = \frac{p_{2s}(1-p_{2s})(3p_{1s}-2)}{p_{1s}(1-p_{1s})(3p_{2s}-2)} \quad (1)$$

However, $p_{2s}(1-p_{2s}) = \frac{1}{2}p_{1s}(p_{1s}-1 + \sqrt{1+2p_{1s}-3p_{1s}^2})$. Combining this with (1),

$$\frac{\zeta_{2s}}{\zeta_{1s}} = \frac{1}{2} \frac{(p_{1s}-1 + \sqrt{1+2p_{1s}-3p_{1s}^2})(3p_{1s}-2)}{(1-p_{1s})(3p_{2s}-2)} = \frac{1}{2} \frac{(\sqrt{1+3p_{1s}} - \sqrt{1-p_{1s}})(3p_{1s}-2)}{\sqrt{1-p_{1s}} \cdot (3p_{2s}-2)} \quad (2)$$

⁷This corresponds to the risk dominant equilibrium (*Hare, Hare*).

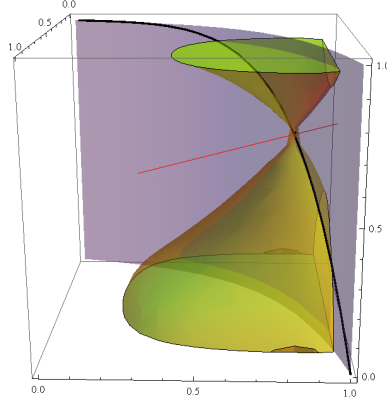


Figure 3: Examples of stable manifolds and invariant sets for star with 3 agents

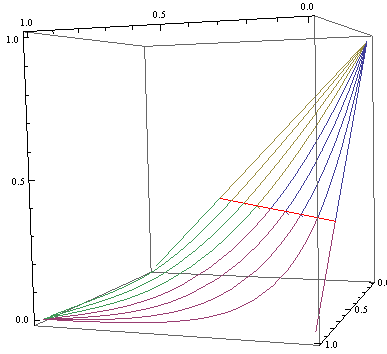


Figure 4: Examples of stable manifolds for star with 3 agents

Similarly, we have that $3p_{2s} - 2 = \frac{1}{2}\sqrt{1+3p_{1s}} \cdot (3\sqrt{1-p_{1s}} - \sqrt{1+3p_{1s}})$. By multiplying and dividing equation (2) with $(\sqrt{1+3p_{1s}} + 3\sqrt{1-p_{1s}})$ we get:

$$\begin{aligned}
\frac{\zeta_{2s}}{\zeta_{1s}} &= \frac{1}{2} \frac{(\sqrt{1+3p_{1s}} + 3\sqrt{1-p_{1s}})(\sqrt{1+3p_{1s}} - \sqrt{1-p_{1s}})(3p_{1s} - 2)}{2\sqrt{1-p_{1s}} \cdot \sqrt{1+3p_{1s}} \cdot (2 - 3p_{1s})} \\
&= -\frac{1}{4} \frac{(\sqrt{1+3p_{1s}} + 3\sqrt{1-p_{1s}})(\sqrt{1+3p_{1s}} - \sqrt{1-p_{1s}})}{\sqrt{1+2p_{1s} - 3p_{1s}^2}} \\
&= \frac{1}{2} \left(-1 + \frac{1 - 3p_{1s}}{\sqrt{1+2p_{1s} - 3p_{1s}^2}} \right) = \frac{\partial \left(\frac{1}{2}(1 - p_{1s} + \sqrt{1+2p_{1s} - 3p_{1s}^2}) \right)}{\partial p_{1s}} = \frac{\partial p_{2s}}{\partial p_{1s}}.
\end{aligned}$$

■

5.2.2 N -player Star Graph

In this subsection we show how to estimate the topology of regions of attraction for network of coordination games. Specifically, we focus on a star topology where each node is an agent and each edge corresponds to our benchmark example of the Stag Hunt game. This game has two pure Nash equilibria where all agents either play *Stag*, or play *Hare*. On top of those the game has a continuum of mixed Nash equilibria. Our goal is to produce an oracle which given as input an initial condition outputs the resulting equilibrium that system converges to.

Example. In order to gain some intuition on the construction of these oracles let's focus on the minimal case with a continuum of equilibria ($n = 3$, center with two neighbors). Since each agent has two strategies it suffices to depict for each one the probability with which they

choose strategy *Hare*. Hence, the phase space can be depicted in 3 dimensions. Figure 4 depicts this phase space. The game exhibits a continuum of unstable fully randomized equilibria (the red line) and two attracting pure equilibria, the one where all agents choose strategy *Stag* and the one where all agents choose strategy *Hare*. For any mixed Nash equilibrium there exists a curve (co-dimension 2) of points that converge to it. Figure 4 has plotted several such stable manifolds for sample mixed equilibria along the equilibrium line. The union of these stable manifolds partitions the state space into two regions, the upper region (attracting for the equilibrium $(Hare, Hare, Hare)$) and the lower one (attracting for the equilibrium $(Stag, Stag, Stag)$). Hence, in order to construct our oracle it suffices to have a description of these attracting curves for the mixed equilibria. However, as shown in figure 3, we have identified two distinct invariant functions for the replicator dynamic in this system. Given any mixed Nash equilibrium, the set of points of the state space which agree with the value of each of these invariant functions define a set of co-dimension one (the double hollow cone and the curved plane). Any points that converge to this equilibrium must lie on the intersection of these sets (black curve). In fact, due to our point-wise convergence theorem, it immediately follows that this intersection is exactly the stable manifold of the unstable equilibrium. The case for general n works analogously, but now we need to identify $n - 1$ invariant functions in an algorithmic, efficient manner.

The details and proof of correctness of the oracle implementation lie in Appendix B.

5.3 Average Price of Anarchy Analysis in Stag Hunt Game via Polytope Approximations of Regions of Attraction

We focus on the following generalized class of Stag Hunt game, as described in figure 1(b). The w parameter is greater or equal to 1⁸. We denote an instance of such a game as $G(w)$. It is straightforward to check that $G(2)$ is equivalent to the standard stag hunt game (modulo payoff shifts to the agents). For $G(w)$, it is straightforward to check that $p_{1s}^w(1 - p_{1s}) - p_{2s}^w(1 - p_{2s})$ is an invariant property of the replicator system. The presence of the parameter w on the exponent precludes the existence of a simple, explicit, parametric description of all the solutions. We analyze the topology of the basins of attractions and produce simple subsets/supersets of them. The volume of these polytope approximations (see figure 4) can be computed explicitly and these measures can be used to provide upper and lower bounds on the average system performance and average price of anarchy.

Theorem 13. *The average price of anarchy of $G(w)$ with $w \geq 1$ is at most $\frac{w^2+w}{w^2+1}$ and at least $\frac{w(w+1)^2}{w(w+1)^2-2w+2}$.*

By combining the exact analysis of the standard Stag Hunt game (theorem 11) and theorem 13 we derive that:

Corollary 14. *The average price of anarchy of the class of $G(w)$ games with $w > 0$ is at least $\frac{2}{1+\frac{9+2\sqrt{3}\pi}{27}} \approx 1.15$ and at most $\frac{4+3\sqrt{2}}{4+2\sqrt{2}} \approx 1.21$. In comparison, the price of anarchy for this class of games is unbounded.*

5.4 Balls and Bins

In this section we consider the classic game of n identical balls, with n identical bins. Each ball chooses a distribution over the bins selfishly and we assume that the cost of bin γ is equal to γ 's load. We know for this game that the PoA is $\Omega(\frac{\log n}{\log \log n})$ [11]. We will prove that the Average PoA is 1. This is derived via corollary 6 and by showing that in this case the set of weakly stable Nash equilibria coincides with the set of pure equilibria.

⁸It is easy to see that for any $0 < w < 1$, $G(w)$ is isomorphic to $G(1/w)$ after relabeling of strategies.

Lemma 15. *In the problem of n identical balls and n identical bins every weakly stable Nash equilibrium is pure.*

From lemma 15 we get that for all but measure zero starting points of $g(\Delta)$, the replicator converges to pure Nash Equilibria. Every pure Nash equilibrium (each ball chooses a distinct bin) has social cost (makespan) 1 which is also the optimal. Hence the Average PoA is 1.

Lemma 16. *Let $n \geq 4$ then the set of Nash equilibria of the n balls n bins game is uncountable.*

5.5 Average PoA $5/2 - \epsilon$?

We prove the following lemma which raises an interesting question for future work.

Lemma 17. *For any congestion game G with linear costs there exists a constant $\epsilon(G)$ so that the Average PoA is at most $\frac{5}{2} - \epsilon$ for replicator dynamics.*

It is reasonable to ask whether or not there exists a global ϵ so that any linear congestion game has APoA at most $\frac{5}{2} - \epsilon$.

6 Conclusion

We have discussed several different novel approaches to analyzing learning dynamics in games, and specifically in games with many equilibria. Our approaches extend the typical outlook of what it means to understand learning in games by pointing out and resolving several important open questions in classic settings, such as evolutionary network coordination games and linear congestion games.

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A Appendix

A.1 Missing proofs

In the presentation of the proofs sometimes it is convenient to represent the flow $\phi(t, p)$ via the map $\phi_t(p)$, where the time variable is used as a parameter.

A.1.1 Proof of theorem 2

Proof We observe that

$$\Psi(\mathbf{p}) = \sum_i \hat{u}_i = \sum_{i,\gamma} p_{i\gamma} \sum_{j \in N(i)} \sum_{\delta} A_{ij}^{\gamma\delta} p_{j\delta}$$

is a Lyapunov function for our game since (strict increasing along the trajectories)

$$\frac{\partial \Psi}{\partial p_{i\gamma}} = u_{i\gamma} + \sum_{j \in N(i)} \sum_{\delta} A_{ji}^{\delta\gamma} p_{j\delta} = 2u_{i\gamma} \text{ since } A_{ij} = A_{ji}^T$$

and hence

$$\begin{aligned} \frac{d\Psi}{dt} &= \sum_{i,\gamma} \frac{\partial \Psi}{\partial p_{i\gamma}} \frac{dp_{i\gamma}}{dt} \\ &= \sum_{i,\gamma,\gamma'} p_{i\gamma} p_{i\gamma'} (u_{i\gamma} - u_{i\gamma'})^2 \geq 0 \end{aligned}$$

with equality at fixed points. Hence (as in [18]) we have convergence to equilibria sets (compact connected sets consisting of fixed points). We address the fact that this doesn't suffice for pointwise convergence. To be exact it suffices only in the case the equilibria are isolated (which is not the case for network coordination games - see figure 4).

Let \mathbf{q} be a limit point of the trajectory $\mathbf{p}(t)$ where $\mathbf{p}(t)$ is in the interior of Δ for all $t \in \mathbb{R}$ (since we started from an initial condition inside Δ) then we have that $\Psi(\mathbf{q}) > \Psi(\mathbf{p}(t))$. We define the relative entropy.

$$I(\mathbf{p}) = - \sum_i \sum_{\gamma: q_{i\gamma} > 0} q_{i\gamma} \ln(p_{i\gamma}/q_{i\gamma}) \geq 0 \text{ (Jensen's ineq.)}$$

and $I(\mathbf{p}) = 0$ iff $\mathbf{p} = \mathbf{q}$. We get that

$$\begin{aligned} \frac{dI}{dt} &= - \sum_i \sum_{\gamma: q_{i\gamma} > 0} q_{i\gamma} (u_{i\gamma} - \hat{u}_i) \\ &= \sum_i \hat{u}_i - \sum_{i,\gamma} q_{i\gamma} u_{i\gamma} \\ &= \sum_i \hat{u}_i - \sum_{i,\gamma} \sum_{j \in N(i)} \sum_{\delta} A_{ij}^{\gamma\delta} p_{j\delta} q_{i\gamma} \\ &= \sum_i \hat{u}_i - \sum_{j,\delta} \sum_{i \in N(j)} \sum_{\gamma} A_{ij}^{\gamma\delta} p_{j\delta} q_{i\gamma} \text{ (since } A_{ij} = A_{ji}^T) \\ &= \sum_i \hat{u}_i - \sum_{j,\delta} p_{j\delta} d_{j\delta} \\ &= \sum_i \hat{u}_i - \sum_i \hat{d}_i - \sum_{j,\delta} p_{j\delta} (d_{j\delta} - \hat{d}_j) \\ &= \Psi(\mathbf{p}) - \Psi(\mathbf{q}) - \sum_{i,\gamma} p_{i\gamma} (d_{i\gamma} - \hat{d}_i) \end{aligned}$$

where $d_{i\gamma}, \hat{d}_i$ correspond to the payoff of player i if he chooses strategy γ and his expected payoff respectively at point \mathbf{q} . The rest of the proof follows in a similar way to Losert and Akin.

We break the term $\sum_{i,\gamma} p_{i\gamma}(d_{i\gamma} - \hat{d}_i)$ to positive and negative terms (we ignore zero terms), i.e., $\sum_{i,\gamma} p_{i\gamma}(d_{i\gamma} - \hat{d}_i) = \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i) + \sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i)$.

Claim: There exists an $\epsilon > 0$ so that the function $Z(\mathbf{p}) = I(\mathbf{p}) + 2 \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}$ has $\frac{dZ}{dt} < 0$ for $|\mathbf{p} - \mathbf{q}| < \epsilon$ and $\Psi(\mathbf{q}) > \Psi(\mathbf{p})$.

Assuming that $\mathbf{p} \rightarrow \mathbf{q}$, we get $u_{i\gamma} - \hat{u}_i \rightarrow d_{i\gamma} - \hat{d}_i$ for all i, γ . Hence for small enough $\epsilon > 0$ with $|\mathbf{p} - \mathbf{q}| < \epsilon$, we have that $u_{i\gamma} - \hat{u}_i \leq \frac{3}{4}(d_{i\gamma} - \hat{d}_i)$ for the terms which $d_{i\gamma} - \hat{d}_i < 0$. Therefore

$$\begin{aligned} \frac{dZ}{dt} &= \Psi(\mathbf{p}) - \Psi(\mathbf{q}) - \sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i) - \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i) + 2 \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(u_{i\gamma} - \hat{u}_i) \\ &\leq \Psi(\mathbf{p}) - \Psi(\mathbf{q}) - \sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i) - \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i) + 3/2 \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i) \\ &= \underbrace{\Psi(\mathbf{p}) - \Psi(\mathbf{q})}_{<0} + \underbrace{\sum_{i,\gamma:\hat{d}_i < d_{i\gamma}} -p_{i\gamma}(d_{i\gamma} - \hat{d}_i)}_{\leq 0} + \underbrace{1/2 \sum_{i,\gamma:\hat{d}_i > d_{i\gamma}} p_{i\gamma}(d_{i\gamma} - \hat{d}_i)}_{\leq 0} < 0 \end{aligned}$$

where we substitute $\frac{p_{i\gamma}}{dt} = p_{i\gamma}(u_{i\gamma} - \hat{u}_i)$ (replicator), and the claim is proved.

Notice that $Z(\mathbf{p}) \geq 0$ (sum of positive terms and $I(\mathbf{p}) \geq 0$) and is zero iff $\mathbf{p} = \mathbf{q}$. (i)

To finish the proof of the theorem, if \mathbf{q} is a limit point of $\mathbf{p}(t)$, there exists an increasing sequence of times t_i , with $t_n \rightarrow \infty$ and $\mathbf{p}(t_n) \rightarrow \mathbf{q}$. We consider ϵ' such that the set $C = \{\mathbf{p} : Z(\mathbf{p}) < \epsilon'\}$ is inside $B = \{\mathbf{p} : |\mathbf{p} - \mathbf{q}| < \epsilon\}$ where ϵ is from claim above. Since $\mathbf{p}(t_n) \rightarrow \mathbf{q}$, consider a time t_N where $\mathbf{p}(t_N)$ is inside C . From claim above we get that $Z(\mathbf{p})$ is decreasing inside B (and hence inside C), thus $Z(\mathbf{p}(t)) \leq Z(\mathbf{p}(t_N)) < \epsilon'$ for all $t \geq t_N$, hence the orbit will remain in C . By the fact that $Z(\mathbf{p}(t))$ is decreasing in C (claim above) and also $Z(\mathbf{p}(t_N)) \rightarrow Z(\mathbf{q}) = 0$ it follows that $Z(\mathbf{p}(t)) \rightarrow 0$ as $t \rightarrow \infty$. Hence $\mathbf{p}(t) \rightarrow \mathbf{q}$ as $t \rightarrow \infty$ using (i). ■

A.1.2 Proof of theorem 4

To prove the theorem we will use Center Stable Manifold Theorem (see Theorem 19). In order to do that we need a map whose domain is full-dimensional. However, a simplex in \mathbb{R}^n has dimension $n - 1$. Therefore, we need to take a projection of the domain space and accordingly redefine the map of the dynamical system. We note that the projection we take will be fixed-point dependent; this is to keep of the proof that every stable fixed point is a weakly stable Nash proved in [18] relatively less involved later. Let \mathbf{q} be a point of our state space Δ and $\Sigma = |\cup_i S_i|$. Let $h_{\mathbf{q}} : [N] \rightarrow [\Sigma]$ be a function such that $h_{\mathbf{q}}(i) = \gamma$ if $q_{i\gamma} > 0$ for some $\gamma \in S_i$ (same definition with discrete case). Let $M = \sum |S_i|$ and g a fixed projection where you exclude the first coordinate of every player's distribution vector. We consider the mapping $z_{\mathbf{q}} : R^M \rightarrow R^{M-N}$ so that we exclude from each player i the variable $p_{i,h_{\mathbf{q}}(i)}$ ($z_{\mathbf{q}}$ plays the same role as g but we drop variables with specific property this time). We substitute the variables $p_{i,h_{\mathbf{q}}(i)}$ with $1 - \sum_{\gamma \in S_i, \gamma \neq h_{\mathbf{q}}(i)} p_{i\gamma}$. Let $\mathcal{J}_{\mathbf{q}}$ be the reduced Jacobian at a fixed point $z_{\mathbf{q}}(\mathbf{q})$.

Definition 18. A fixed point $\mathbf{q} \in \Delta \subset \mathbb{R}^M$ is called unstable if $\mathcal{J}_{\mathbf{q}}$ has an eigenvalue with positive real part. Otherwise is called stable.

The Center/Stable manifold theorem which is of great importance in dynamical systems is formally the following:

Theorem 19. (Center and Stable Manifolds, p. 65 of [35]) Let $\mathbf{0}$ be a fixed point for the C^r local diffeomorphism $f : U \rightarrow \mathbb{R}^n$ where $U \subset \mathbb{R}^n$ is a neighborhood of zero in \mathbb{R}^n and $r \geq 1$. Let

$E^s \oplus E^c \oplus E^u$ be the invariant splitting of \mathbb{R}^n into generalized eigenspaces of $Df(\mathbf{0})$ corresponding to eigenvalues of absolute value less than one, equal to one, and greater than one. To the $Df(\mathbf{0})$ invariant subspace $E^s \oplus E^c$ there is associated a local f invariant C^r embedded disc W_{loc}^{sc} tangent to the linear subspace at $\mathbf{0}$ and a ball B around zero such that:

$$f(W_{loc}^{sc}) \cap B \subset W_{loc}^{sc}. \text{ If } f^n(\mathbf{x}) \in B \text{ for all } n \geq 0, \text{ then } \mathbf{x} \in W_{loc}^{sc} \quad (3)$$

For $t = 1$ and an unstable fixed point \mathbf{p} we consider the function $\psi_{1,\mathbf{p}}(\mathbf{x}) = z_{\mathbf{p}} \circ \phi_1 \circ z_{\mathbf{p}}^{-1}(\mathbf{x})$ which is C^1 local diffeomorphism. Let $B_{z_{\mathbf{p}}(\mathbf{p})}$ be the ball that is derived from 19 and we consider the union of these balls (transformed in \mathbb{R}^M)

$$A = \cup_{\mathbf{p}} A_{z_{\mathbf{p}}(\mathbf{p})}$$

where $A_{z_{\mathbf{p}}(\mathbf{p})} = g \circ z_{\mathbf{p}}^{-1}(B_{z_{\mathbf{p}}(\mathbf{p})})$ ($z_{\mathbf{p}}^{-1}$ "returns" the set $B_{z_{\mathbf{p}}(\mathbf{p})}$ back to \mathbb{R}^M). Taking advantage of separability of \mathcal{R}^M we have the following theorem.

Theorem 20. (Lindelöf's lemma) For every open cover there is a countable subcover.

Therefore we can find a countable subcover for $A = \cup_{\mathbf{p}} A_{z_{\mathbf{p}}(\mathbf{p})}$, i.e., $A = \cup_{m=1}^{\infty} A_{z_{\mathbf{p}_m}(\mathbf{p}_m)}$.

Let $\psi_{n,\mathbf{p}}(\mathbf{x}) = z_{\mathbf{p}} \circ \phi_n \circ z_{\mathbf{p}}^{-1}(\mathbf{x})$. If a point $\mathbf{x} \in \text{int } g(\Delta)$ (which corresponds to $g^{-1}(\mathbf{x})$ in our original Δ) has as unstable fixed point as a limit, there must exist a n_0 and m so that $\psi_{n,\mathbf{p}_m} \circ z_{\mathbf{p}_m} \circ g^{-1}(\mathbf{x}) \in B_{z_{\mathbf{p}_m}(\mathbf{p}_m)}$ for all $n \geq n_0$ and therefore again from 19 we get that $\psi_{n_0,\mathbf{p}_m} \circ z_{\mathbf{p}_m} \circ g^{-1}(\mathbf{x}) \in W_{loc}^{sc} z_{\mathbf{p}_m}(\mathbf{p}_m)$, hence $\mathbf{x} \in g \circ z_{\mathbf{p}_m}^{-1} \circ \psi_{n_0,\mathbf{p}_m}^{-1}(W_{loc}^{sc} z_{\mathbf{p}_m}(\mathbf{p}_m))$.

Hence the set of points in $\text{int } g(\Delta)$ whose ω -limit has an unstable equilibrium is a subset of

$$C = \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} g \circ z_{\mathbf{p}_m}^{-1} \circ \psi_{n,\mathbf{p}_m}^{-1}(W_{loc}^{sc} z_{\mathbf{p}_m}(\mathbf{p}_m)) \quad (4)$$

Observe that the dimension of $W_{loc}^{sc} z_{\mathbf{p}_m}(\mathbf{p}_m)$ is at most $M - N - 1$ since we assume that \mathbf{p}_m is unstable ($\mathcal{J}_{\mathbf{p}_m}$ has an eigenvalue with positive real part)⁹ and thus $\dim E^u \geq 1$, hence the manifold has Lebesgue measure zero in \mathbb{R}^{M-N} . Finally since $g \circ z_{\mathbf{p}_m}^{-1} \circ \psi_{n,\mathbf{p}_m}^{-1} : \mathbb{R}^{M-N} \rightarrow \mathbb{R}^{M-N}$ is continuously differentiable (ψ_{n,\mathbf{p}_m} is a C^1 it is locally Lipschitz (see [27] p.71) and it preserves the null-sets (see lemma 21). Namely, C is a countable union of measure zero sets, i.e., is measure zero as well and the theorem 4 follows¹⁰.

Lemma 21. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a locally Lipschitz function, then g is null-set preserving, i.e., for $E \subset \mathbb{R}^n$ if E has measure zero then $g(E)$ has also measure zero.

Proof Let B_{γ} be an open ball such that $\|g(\mathbf{y}) - g(\mathbf{x})\| \leq K_{\gamma} \|\mathbf{y} - \mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in B_{\gamma}$. We consider the union $\cup_{\gamma} B_{\gamma}$ which cover \mathbb{R}^n by the assumption that g is locally lipschitz. By Lindelöf's lemma we have a countable subcover, i.e., $\cup_{i=1}^{\infty} B_i$. Let $E_i = E \cap B_i$. We will prove that $g(E_i)$ has measure zero. Fix an $\epsilon > 0$. Since $E_i \subset E$, we have that E_i has measure zero, hence we can find a countable cover of open balls C_1, C_2, \dots for E_i , namely $E_i \subset \cup_{j=1}^{\infty} C_j$ so that $C_j \subset B_i$ for all j and also $\sum_{j=1}^{\infty} \mu(C_j) < \frac{\epsilon}{K_i^n}$. Since $E_i \subset \cup_{j=1}^{\infty} C_j$ we get that $g(E_i) \subset \cup_{j=1}^{\infty} g(C_j)$, namely $g(C_1), g(C_2), \dots$ cover $g(E_i)$ and also $g(C_j) \subset g(B_i)$ for all j . Assuming that ball $C_j \equiv B(\mathbf{x}, r)$ (center \mathbf{x} and radius r) then it is clear that $g(C_j) \subset B(g(\mathbf{x}), K_i r)$ (g maps the center \mathbf{x} to $g(\mathbf{x})$ and the radius r to $K_i r$ because of lipschitz assumption). But $\mu(B(g(\mathbf{x}), K_i r)) = K_i^n \mu(B(\mathbf{x}, r)) = K_i^n \mu(C_j)$, therefore $\mu(g(C_j)) \leq K_i^n \mu(C_j)$ and so we conclude that

$$\mu(g(E_i)) \leq \sum_{j=1}^{\infty} \mu(g(C_j)) \leq K_i^n \sum_{j=1}^{\infty} \mu(C_j) < \epsilon$$

⁹Here we used the fact that the eigenvalues with absolute value less than one, one and greater than one of e^A correspond to eigenvalues with negative real part, zero real part and positive real part respectively of A

¹⁰we used "silently" that $\mu(g(\Delta)) = \mu(\text{int } g(\Delta))$

Since ϵ was arbitrary, it follows that $\mu(g(E_i)) = 0$. To finish the proof, observe that $g(E) = \bigcup_{i=1}^{\infty} g(E_i)$ therefore $\mu(g(E)) \leq \sum_{i=1}^{\infty} \mu(g(E_i)) = 0$. ■

A.1.3 Proof of lemma 8

Proof Let i, j two randomized agents so that $(i, j) \in E(G)$. Assume that $0 < p_{i\gamma} < 1$ where \mathbf{p} is a Nash equilibrium and $\{\delta_1, \dots, \delta_m\}$ be the support of strategy of player j with $m > 1$. By the assumption that every row and column has different entries, if i chooses to play γ with probability 1, we get that j will increase her payoff by choosing the strategy $\delta = \arg \max_{\delta_k} A_{ij}^{\gamma \delta_k}$ (unique δ). Hence either i or j must choose a pure strategy to have a weakly stable Nash equilibrium. ■

A.1.4 Proof of lemma 9

Proof Clearly $\psi(\mathbf{x}) = g(\lim_{n \rightarrow \infty} \phi_n(g^{-1}(\mathbf{x}))) = \lim_{n \rightarrow \infty} g \circ \phi_n \circ g^{-1}(\mathbf{x})$ where n is over positive integers (since we have always convergence). For an arbitrary $c \in \mathbb{R}$ we have that

$$\{\mathbf{x} : \psi(\mathbf{x})_i < c\} = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n>m}^{\infty} \{\mathbf{x} : g \circ \phi_n \circ g^{-1}(\mathbf{x})_i < c - \frac{1}{k}\}$$

The set $\{\mathbf{x} : g \circ \phi_n \circ g^{-1}(\mathbf{x})_i < c - \frac{1}{k}\}$ is measurable since $g \circ \phi_n \circ g^{-1}(\mathbf{x})_i$ is measurable, by continuity of g, g^{-1}, ϕ (so continuous is the composition). Therefore $\psi(\mathbf{x})_i$ is a measurable function and so it is $\psi(\mathbf{x})$. ■

A.1.5 Proof of theorem 13

For any w , $G(w)$ is a coordination/potential game and therefore it is payoff equivalent to a congestion game. The only two weakly stable equilibria are the pure ones, hence in order to understand the average case system performance it suffices to understand the size of regions of attraction for each of them. As in the case of Stag Hunt game, we focus on the projection of the system to the subspace $(p_{1s}, p_{2s}) \subset [0, 1]^2$.

We denote by ζ, ψ , the projected flow and vector field respectively.

Lemma 22. *All but a zero measure of initial conditions in the polytope (P_{Hare}) :*

$$\begin{aligned} p_{2s} &\leq -wp_{1s} + w \\ p_{2s} &\leq -\frac{1}{w}p_{1s} + 1 \\ 0 &\leq p_{1s}, p_{2s} \leq 1 \end{aligned}$$

converges to the $(Hare, Hare)$ equilibrium. All but a zero measure of initial conditions in the polytope (P_{Stag}) :

$$\begin{aligned} p_{2s} &\geq -p_{1s} + \frac{2w}{w+1} \\ 0 &\leq p_{1s}, p_{2s} \leq 1 \end{aligned}$$

converges to the $(Stag, Stag)$ equilibrium.

Proof First, we will prove the claimed property for polytope (P_{Stag}) . Since the game is symmetric, the replicator dynamics are similarly symmetric with $p_{2s} = p_{1s}$ axis of symmetry. Therefore it suffices to prove the property for the polytope $P'_{Hare} = P_{Hare} \cap \{p_{2s} \leq p_{1s}\} = \{p_{2s} \leq p_{1s}\} \cap \{p_{2s} \leq -wp_{1s} + w\} \cap \{0 \leq p_{1s} \leq 1\} \cap \{0 \leq p_{2s} \leq 1\}$ We will argue that this polytope is

forward flow invariant, *i.e.*, if we start from an initial condition $\mathbf{x} \in P'_{Hare}$ $\psi(t, \mathbf{x}) \in P'_{Hare}$ for all $t > 0$. On the p_{1s}, p_{2s} subspace P'_{Hare} defines a triangle with vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (\frac{w}{w+1}, \frac{w}{w+1})$ (see figure 4). The line segments AB , AC are trivially flow invariant. Hence, in order to argue that the ABC triangle is forward flow invariant, it suffices to show that everywhere along the line segment BC the vector field does not point “outwards” of the ABC triangle. Specifically, we need to show that for every point p on the line segment BC (except the Nash equilibrium C), $\frac{|\zeta_{1s}(\mathbf{p})|}{|\zeta_{2s}(\mathbf{p})|} \geq \frac{1}{w}$.

$$\frac{|\zeta_{1s}(\mathbf{p})|}{|\zeta_{2s}(\mathbf{p})|} = \frac{p_{1s}|p_{2s} - (p_{1s}p_{2s} + w(1 - p_{1s})(1 - p_{2s}))|}{p_{2s}|p_{1s} - (p_{1s}p_{2s} + w(1 - p_{1s})(1 - p_{2s}))|} = \frac{p_{1s}(1 - p_{1s})(w - (w + 1)p_{2s})}{p_{2s}(1 - p_{2s})(-w + (w + 1)p_{1s})}$$

However, the points of the line passing through B, C satisfy $p_{2s} = w(1 - p_{1s})$.

$$\begin{aligned} \frac{|\zeta_{1s}(\mathbf{p})|}{|\zeta_{2s}(\mathbf{p})|} &= \frac{wp_{1s}(1 - p_{1s})(1 - (w + 1)(1 - p_{1s}))}{w(1 - p_{1s})(1 - w(1 - p_{1s}))(-w + (w + 1)p_{1s})} \\ &= \frac{p_{1s}(-w + (w + 1)p_{1s})}{(1 - w + wp_{1s})(-w + (w + 1)p_{1s})} \\ &= \frac{p_{1s}}{1 - w + wp_{1s}} \geq \frac{p_{1s}}{wp_{1s}} = \frac{1}{w} \end{aligned}$$

We have established that the ABC triangle is forward flow invariant. Since $G(w)$ is a potential game, all but a zero measurable set of initial conditions converge to one of the two pure equilibria. Since ABC is forward invariant, all but a zero measure of initial conditions converge to $(Hare, Hare)$. A symmetric argument holds for the triangle $AB'C$ with $B' = (0, 1)$. The union of ABC and $AB'C$ is equal to the polygon P_{Hare} , which implies the first part of the lemma.

Next, we will prove the claimed property for polytope (P_{Stag}) . Again, due to symmetry, it suffices to prove the property for the polytope $P'_{Stag} = P_{Stag} \cap \{p_{2s} \leq p_{1s}\} = \{p_{2s} \leq p_{1s}\} \cap \{p_{2s} \geq -p_{1s} + \frac{2w}{w+1}\} \cap \{0 \leq p_{1s} \leq 1\} \cap \{0 \leq p_{2s} \leq 1\}$. We will argue that this polytope is forward flow invariant. On the p_{1s}, p_{2s} subspace P'_{Stag} defines a triangle with vertices $D = (1, \frac{w-1}{w+1})$, $E = (1, 1)$ and $C = (\frac{w}{w+1}, \frac{w}{w+1})$. The line segments CD , DE are trivially forward flow invariant. Hence, in order to argue that the CDE triangle is forward flow invariant, it suffices to show that everywhere along the line segment CD the vector field does not point “outwards” of the CDE triangle (see figure 4). Specifically, we need to show that for every point p on the line segment CD (except the Nash equilibrium C), $\frac{|\zeta_{1s}(\mathbf{p})|}{|\zeta_{2s}(\mathbf{p})|} \leq 1$.

$$\frac{|\zeta_{1s}(\mathbf{p})|}{|\zeta_{2s}(\mathbf{p})|} = \frac{p_{1s}|p_{2s} - (p_{1s}p_{2s} + w(1 - p_{1s})(1 - p_{2s}))|}{p_{2s}|p_{1s} - (p_{1s}p_{2s} + w(1 - p_{1s})(1 - p_{2s}))|} = \frac{p_{1s}(1 - p_{1s})(w - (w + 1)p_{2s})}{p_{2s}(1 - p_{2s})(-w + (w + 1)p_{1s})}$$

However, the points of the line passing through C, D satisfy $p_{2s} = -p_{1s} + \frac{2w}{w+1}$.

$$\begin{aligned} \frac{|\zeta_{1s}(\mathbf{p})|}{|\zeta_{2s}(\mathbf{p})|} &= \frac{p_{1s}(1 - p_{1s})(-w + (w + 1)p_{1s})}{(-p_{1s} + \frac{2w}{w+1})(-\frac{w-1}{w+1} + p_{1s})(-w + (w + 1)p_{1s})} \\ &= \frac{p_{1s}(1 - p_{1s})}{(-p_{1s} + \frac{2w}{w+1})(-\frac{w-1}{w+1} + p_{1s})} = \frac{p_{1s}(1 - p_{1s})}{\frac{2(w-1)}{w+1}(-\frac{w}{w+1} + p_{1s}) + p_{1s}(1 - p_{1s})} \leq 1 \end{aligned}$$

We have established that the CDE triangle is forward flow invariant. Since $G(w)$ is a potential game, all but a zero measurable set of initial conditions converge to one of the two pure equilibria. Since CDE is forward invariant, all but a zero measure of initial conditions converge

to $(Stag, Stag)$. A symmetric argument holds for the triangle $CD'E$ with $D' = (\frac{w-1}{w+1}, 1)$. The union of CDE and $CD'E$ is equal to the polygon P_{Stag} , which implies the second part of the lemma. ■

Proof The measure/size of $\mu(P_{Hare}) = 2|ABC| = \frac{w}{w+1}$, and similarly the measure of $\mu(P_{Stag}) = 2|CDE| = \frac{2}{(w+1)^2}$. The average limit performance of the replicator satisfies $\int_{g(\Delta)} sw(\psi(x))d\mu \geq 2w \cdot \mu(P_{Hare}) + 2(1 - \mu(P_{Hare})) = 2\frac{w^2+1}{w+1}$. Furthermore, $\int_{g(\Delta)} sw(\psi(x))d\mu \leq 2w(1 - \mu(P_{Stag})) + 2 \cdot \mu(P_{Stag}) = 2w(1 - \frac{2}{(w+1)^2}) + 2 \cdot \frac{2}{(w+1)^2} = 2w - 4\frac{w-1}{(w+1)^2}$. This implies that $\frac{w(w+1)^2}{w(w+1)^2 - 2w + 2} \leq APoA \leq \frac{w^2+w}{w^2+1}$. ■

A.1.6 Proof of lemma 15

Proof Assume we have a weakly Nash equilibrium \mathbf{p} . From corollary 6, we have the following facts:

- Fact 1: For every bin γ , if a player i chooses γ with probability $1 > p_{i\gamma} > 0$, he must be the only player that chooses that bin with nonzero probability. Let i, j two players that choose bin γ with nonzero probabilities and also $p_{i\gamma}, p_{j\gamma} < 1$. Clearly if player i changes his strategy and chooses bin γ with probability one, then player j doesn't stay indifferent (his cost $c_{i\gamma}$ increases).
- Fact 2: If player i chooses bin γ with probability one, then he is the only player that chooses bin γ with nonzero probability. This is true because every player $j \neq i$ can find a bin with load less than 1 to choose.

From Facts 1,2 and since the number of balls is equal to the number of bins we get that \mathbf{p} must be pure. ■

A.1.7 Proof of lemma 16

Proof We will prove it for $n = 4$ and the generalization is then easy, i.e., if $n > 4$ then the first 4 players will play as shown below in the first 4 bins and each of the remaining $n - 4$ players will choose a distinct remaining bin. Below we give matrix A where $A_{i\gamma} = p_{i\gamma}$. Observe that for any $x \in [\frac{1}{4}, \frac{3}{4}]$ we have a Nash equilibrium.

$$A = \begin{pmatrix} x & 1-x & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & x & 1-x \end{pmatrix}$$

■

A.1.8 Proof of lemma 17

Proof We consider among all the pure Nash equilibrium of the game G , one \mathbf{q} with minimum Φ . Since for all pure strategies \mathbf{s} we have that $\Phi(\mathbf{s}) \leq sc(\mathbf{s}) \leq 2\Phi(\mathbf{s})$, the following inequalities holds for all mixed strategies \mathbf{p}

$$\Psi(\mathbf{p}) \leq sc(\mathbf{p}) \leq 2\Psi(\mathbf{p}) \tag{5}$$

$$sc(\mathbf{q}) \leq 2\Phi(\mathbf{q}) = 2\Psi(\mathbf{q}) \leq 2\Psi(\mathbf{p}) \leq 2sc(\mathbf{p}) \tag{6}$$

We consider the open interval $U = (0, 1.24 \cdot \Psi(\mathbf{q}))$. Hence the set $V = g(\Psi^{-1}(U))$ must be open since $\Psi \circ g^{-1}$ is continuous. Finally $(\text{int } g(\Delta)) \cap V$ is open (intersection of open sets) and nonempty, hence it has positive measure $\epsilon' > 0$ in \mathbb{R}^{M-N} . Assume that the minimum social

cost has value opt and also by 5 it follows that $\Psi(\mathbf{q}) \leq opt$ (since $\Psi(\mathbf{q}) \leq sc(\mathbf{p})$ for all mixed strategies \mathbf{p}).

For all $\mathbf{x} \in V$ we have that $\Psi(g^{-1}(\mathbf{x})) \leq 1.24\Psi(\mathbf{q})$ and the fact that Ψ is decreasing (w.r.t time) we get that ω -limit point of \mathbf{x} will have potential at most $1.24\Psi(\mathbf{q})$, therefore social cost at most $2.48\Psi(\mathbf{q})$ (using 5), thus at most $2.48opt$. For all $\mathbf{x} \in \text{int } g(\Delta)$, $\mathbf{x} \notin V$, the limit point of \mathbf{x} will be a Nash equilibrium with cost at most $\frac{5}{2}opt$ (PoA $\leq 5/2$ in [9], [32]) Therefore the Average PoA is at most

$$\frac{2.48opt \cdot \epsilon' + 2.5opt \cdot (\mu(g(\Delta)) - \epsilon')}{opt \cdot \mu(g(\Delta))} = 2.5 - \frac{0.02\epsilon'}{\mu(g(\Delta))}$$

■

B N-star graph

Here is the high level idea of the analysis: We start the analysis by showing that the only fixed points with region of attraction with positive measure are when all players choose strategy *Stag* or all players choose strategy *Hare*. After that we show that the limit point will be either one of the two mentioned, or a fully mixed. Therefore we need to compute the regions of attraction of the 2 fixed points where all choose *Stag* or all choose *Hare*. To do that, we need to compute the boundary of these two regions (namely the center/stable manifold of the fully mixed ones). This happens as follows: Given an initial point (x_1, \dots, x_n, y) , we compute the possible fully mixed limit point $(x'_1, \dots, x'_n, \frac{w}{w+1})$ (will be one possible because we have one variable of freedom due to lemma 24 below) that is on the boundary of the two regions. If the initial condition is on the upper half space w.r.t to the possible fully mixed limit point $(x'_1, \dots, x'_n, \frac{w}{w+1})$ the dynamics converge to the everyone playing *Stag*, otherwise to everyone playing *Hare*.

To simplify notation in the remainder of this section, we rename strategy *Stag* as strategy *A* and strategy *Hare* as strategy *B*.

B.0.9 Structure of fixed points

If a "leaf" i player has a mixed strategy, it must be the case that $z^A = \frac{w}{w+1}$ because $u_{iA} = u_{iB}$ and $u_{iA} = z^A$ and $u_{iB} = w(1 - z^A)$ (u_{iA}, u_{iB} are the expected utilities for player i given that he chooses A, B respectively with probability 1). Hence the fixed points of the star graph game have the following structure: If the middle player has a pure strategy, then all players must be pure. If the middle player has a mixed strategy then $\sum_i p_i^A = \frac{w}{w+1}n$. In that case, if all the "leaf" players have pure strategies then z^A can have any value in $[0, 1]$, otherwise $z^A = \frac{w}{w+1}$.

Lemma 23. *Given a star-graph, the only stable fixed points are those which all players choose strategy A or all players choose strategy B.*

Proof The stable fixed points must be weakly stable Nash equilibria (lemma 5). If at least one "leaf" player is randomized, then from lemma 8 the middle player is pure. Then all the leaf players must play the same strategy as the middle player to have a Nash equilibrium.

In case the middle player is randomized (let $p, 1 - p$ be the probability that he chooses A, B in the fixed point) and all the "leaf" players are pure, then the probabilities of all the "leaf" players to choose strategy A as functions have the same monotonicity properties (they are increasing/decreasing in the same time intervals). They are all increasing in the intervals where $p(t) \geq \frac{w}{w+1}$ and decreasing if $p(t) \leq \frac{w}{w+1}$, where $p(t)$ is a trajectory so that $p(t) \rightarrow p$. Hence all the "leaf" players must agree in the limit, and so the middle player must do to have a stable fixed point. ■

Let $p_i^A, p_i^B = 1 - p_i^A$ be the probability that player i chooses A, B respectively. We use the letter z for the middle player (i.e $z^A, z^B = 1 - z^A$).

B.0.10 Invariants

Lemma 24. $[\ln(p_i^A(t)) - \ln(p_i^B(t))] - [\ln(p_j^A(t)) - \ln(p_j^B(t))]$ is invariant for all i, j (independent of t).

Proof $\frac{d}{dt}[\ln(p_i^A(t)) - \ln(p_i^B(t))] - [\ln(p_j^A(t)) - \ln(p_j^B(t))] = [z^A - w \cdot z^B] - [z^A - w \cdot z^B] = 0$ ■

Lemma 25. For all initial conditions in the interior of Δ , either the dynamic converges to all A 's or all B 's or some fully mixed fixed point.

Proof We consider the following two cases:

- If $p_i^A(t) \rightarrow 1$ for some i , then $\ln[p_i^A(t)] - \ln[p_i^B(t)] \rightarrow +\infty$. So from lemma 24 for every j we get that $\ln[p_j^A(t)] - \ln[p_j^B(t)] \rightarrow +\infty$, hence $p_j^A(t) \rightarrow 1$. Since $z^A(t) \rightarrow 0$ or 1 (because of the structure of the fixed points), and the fact that $p_i^A = 1$ and $z^A = 0$ is repelling we get that the system converges to all A 's. The same argument is used if $p_i^A(t) \rightarrow 0$ for some i .
- If the dynamic converges to a state where all "leaf" players are mixed, then $z^A = \frac{w}{w+1}$ because of the structure of fixed points. ■

Since we assume that we are in the interior of Δ , apart from states all A 's or all B 's, the system can converge to fixed points which have $z^A = \frac{w}{w+1}$ and $\sum p_i^A = \frac{w}{w+1}n$.

Let (x_1, \dots, x_n, y) be the initial condition, where x_i, y are the probabilities players i , middle choose A ($1 - x_i, 1 - y$ will be the probability to choose B) respectively. Using lemma 24 we can find the fixed point (x'_1, \dots, x'_n, y') that the trajectory can converge to apart from the all A 's or all B 's. Suppose that we compute the positive constant c_i for which

$$x_i/(1 - x_i) = c_i x_1/(1 - x_1)$$

The fixed point that the trajectory can possibly converge satisfies $y' = \frac{w}{w+1}$, $x'_i/(1 - x'_i) = c_i x'_1/(1 - x'_1)$ (by subsection B.0.9, lemma 24) and $\sum_i x'_i = \frac{w}{w+1}n$. Therefore we get that

$$x'_i = \frac{c_i x'_1}{1 + (c_i - 1)x'_1} \quad (7)$$

$$x'_1 \left(\sum \frac{c_i}{1 + (c_i - 1)x'_1} \right) = \frac{w}{w+1}n \quad (8)$$

Observe that the function $f(x) = \frac{cx}{1+(c-1)x}$ is strictly increasing in $[0, 1]$ (c any fixed positive) and $f(0) = 0, f(1) = 1$. Therefore $g(x) = \sum \frac{c_i x}{1+(c_i-1)x} - \frac{w}{w+1}n$ is strictly increasing in $[0, 1]$ (as sum of strictly increasing functions in $[0, 1]$) and $g(0) = -\frac{w}{w+1}n < 0$ and $g(1) = n - \frac{w}{w+1}n > 0$, namely it has always a unique solution in $[0, 1]$. Therefore the system of equations has a unique solution, together with $y' = \frac{w}{w+1}$ lie inside Δ . Given x_1, \dots, x_n we can compute (approximate with arbitrary small error ϵ) x'_1, \dots, x'_n via binary search (make use of Bolzano's theorem).

Lemma 26. Since star graph is a bipartite graph from lemma 30 we have that

$$\frac{w}{w+1} \ln(z^A(t)) + \frac{1}{w+1} \ln(z^B(t)) - \sum_i [x'_i \ln(p_i^A(t)) + (1 - x'_i) \ln(p_i^B(t))]$$

is invariant, i.e. independent of t .

Lemma 27. *if $z^A(t) > \frac{w}{w+1}$ and $\sum p_i^A(t) > \frac{w}{w+1}n$ for some t then the trajectory converges to all A's and if $z^A(t) < \frac{w}{w+1}$ and $\sum p_i^A(t) < \frac{w}{w+1}n$ for some t the trajectory converges to all B's.*

Proof This is quite easy. Both $z^A(t)$, $p_i^A(t)$ (for all i) are increasing and also $z^A(t') > \frac{w}{w+1}$ and $\sum p_i^A(t') > \frac{w}{w+1}n$ keep holding for $t' \geq t$ in the first case. In the second case $z^A(t)$, $p_i^A(t)$ (for all i) are decreasing and also $z^A(t') < \frac{w}{w+1}$ and $\sum p_i^A(t') < \frac{w}{w+1}n$ keep holding for $t' \geq t$. ■

Therefore if a trajectory converges to $x'_1, \dots, x'_n, \frac{w}{w+1}$ point then at any time we have $\sum p_i^A(t) > \frac{w}{w+1}n$ and $z^A(t) < \frac{w}{w+1}$ ($p_1^A(t), \dots, p_n^A(t)$ are decreasing and $z^A(t)$ increasing) or $\sum p_i^A(t) < \frac{w}{w+1}n$ and $z^A(t) > \frac{w}{w+1}$ ($p_1^A(t), \dots, p_n^A(t)$ are increasing and $z^A(t)$ decreasing). Combining all the facts together, we get that the stable manifold of the fixed point $(x'_1, \dots, x'_n, \frac{w}{w+1})$ can be described as follows: (x_1, \dots, x_n, y) lies on the stable manifold if $\sum_i x_i > n\frac{w}{w+1}$ and $y < \frac{w}{w+1}$ or $\sum_i x_i < n\frac{w}{w+1}$ and $y > \frac{w}{w+1}$ and by lemma 26 we get that $y^{\frac{w}{w+1}}(1-y)^{\frac{1}{w+1}} = c \prod_i x_i^{x'_i}(1-x_i)^{1-x'_i}$ where $c = \frac{(\frac{w}{w+1})^{\frac{w}{w+1}}(\frac{1}{w+1})^{\frac{1}{w+1}}}{\prod_i (x'_i)^{x'_i}(1-x'_i)^{1-x'_i}}$.

Lemma 28. *The function $x^w(1-x)$ is strictly increasing in $[0, \frac{w}{w+1}]$ and decreasing in $[\frac{w}{w+1}, 1]$.*

Using lemma 28 we have the oracle below:

B.0.11 Oracle Algorithm

Oracle

1. Input: (x_1, \dots, x_n, y)
2. Output: A or B or mixed
3. if $\sum x_i > \frac{w}{w+1}n$ and $y > \frac{w}{w+1}$ return A.
4. if $\sum x_i < \frac{w}{w+1}n$ and $y < \frac{w}{w+1}$ return B.
5. Compute by solving system 7-8 x'_1, \dots, x'_n (binary search)
6. Let $f(t) = \left(\frac{t(w+1)}{w}\right)^{\frac{w}{w+1}} [(1-t)(w+1)]^{\frac{1}{w+1}} - \prod_i \left(\frac{x_i}{x'_i}\right)^{x'_i} \left(\frac{1-x_i}{1-x'_i}\right)^{1-x'_i}$
7. if $(\sum_i x_i > \frac{w}{w+1}n$ and $f(y) < 0)$ or $(\sum_i x_i < \frac{w}{w+1}n$ and $f(y) > 0)$ return B.
8. if $(\sum_i x_i > \frac{w}{w+1}n$ and $f(y) > 0)$ or $(\sum_i x_i < \frac{w}{w+1}n$ and $f(y) < 0)$ return A.
9. return mixed

Remark: Given any point from Δ uniformly at random, under the assumption of solving exactly the equations to compute x'_1, \dots, x'_n and infinite precision the probability that the oracle above returns mixed is zero.

C Invariants

Finding bounds or computing average PoA is not an easy task in most of the cases. Computing average PoA essentially means to learn the probability distribution over the fixed points (as it occurs by the dynamics, in point-wise convergent systems), i.e compute the regions of attraction of the fixed points.

Lemma 29. [16] If \mathbf{q} is an asymptotically stable equilibrium point for a system $\dot{x} = f(x)$ where $f \in C^1$, then its region of attraction $R_{\mathbf{q}}$ is an invariant set whose boundaries are formed by trajectories.

The trajectories that consist the boundaries are the center/stable manifolds of the fixed points. These are sets which are also flow invariant (if the initial condition belongs in this set, the whole trajectory remains in the set). The following theorem, a modification of which can also be found here [29, 28] for zero sum games, gives us invariants of the system.

Lemma 30. Let $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_N(t))$ with $\mathbf{p}(0) \in \Delta$ be a trajectory of the replicator dynamic when applied to a bipartite network of coordination games that has a fully mixed Nash equilibrium $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ then $\sum_{i \in V_{left}} H(\mathbf{q}_i, \mathbf{p}_i(t)) - \sum_{i \in V_{right}} H(\mathbf{q}_i, \mathbf{p}_i(t))$ is invariant.

Proof The derivative of $\sum_{i \in V_{left}} \sum_{\gamma \in S_i} q_{i\gamma} \cdot \ln(p_{i\gamma}) - \sum_{i \in V_{right}} \sum_{\gamma \in S_i} q_{i\gamma} \cdot \ln(p_{i\gamma})$ has as follows:

$$\begin{aligned}
& \sum_{i \in V_{left}} \sum_{\gamma \in S} q_{i\gamma} \frac{d \ln(p_{i\gamma})}{dt} - \sum_{i \in V_{right}} \sum_{\gamma \in S} q_{i\gamma} \frac{d \ln(p_{i\gamma})}{dt} = \sum_{i \in V_{left}} \sum_{\gamma \in S} q_{i\gamma} \frac{\dot{p}_{i\gamma}}{p_{i\gamma}} - \sum_{i \in V_{right}} \sum_{\gamma \in S} q_{i\gamma} \frac{\dot{p}_{i\gamma}}{p_{i\gamma}} = \\
& = \sum_{i \in V_{left}} \sum_{(i,j) \in E} (\mathbf{q}_i^T A_{ij} \mathbf{p}_j - \mathbf{p}_i^T A_{ij} \mathbf{p}_j) - \sum_{i \in V_{right}} \sum_{(i,j) \in E} (\mathbf{q}_i^T A_{ij} \mathbf{p}_j - \mathbf{p}_i^T A_{ij} \mathbf{p}_j) = \\
& = \sum_{i \in V_{left}} \sum_{(i,j) \in E} (\mathbf{q}_i^T - \mathbf{p}_i^T) A_{ij} \mathbf{p}_j - \sum_{i \in V_{right}} \sum_{(i,j) \in E} (\mathbf{q}_i^T - \mathbf{p}_i^T) A_{ij} \mathbf{p}_j = \\
& = \sum_{i \in V_{left}} \sum_{(i,j) \in E} (\mathbf{q}_i^T - \mathbf{p}_i^T) A_{ij} (\mathbf{p}_j - \mathbf{q}_j) - \sum_{i \in V_{right}} \sum_{(i,j) \in E} (\mathbf{q}_i^T - \mathbf{p}_i^T) A_{ij} (\mathbf{p}_j - \mathbf{q}_j) = \\
& = - \sum_{(i,j) \in E, i \in V_{left}, j \in V_{right}} [(\mathbf{q}_i^T - \mathbf{p}_i^T) A_{ij} (\mathbf{q}_j - \mathbf{p}_j) - (\mathbf{q}_j^T - \mathbf{p}_j^T) A_{ji} (\mathbf{q}_i - \mathbf{p}_i)] = 0
\end{aligned}$$

■

The cross entropy between the Nash \mathbf{q} and the state of the system, however is equal to the summation of the K-L divergence between these two distributions and the entropy of \mathbf{q} . Since the entropy of \mathbf{q} is constant, we derive the following corollary (rephrasing previous lemma):

Corollary 31. Let $\mathbf{p}(t)$ with $\mathbf{p}(0) \in \Delta$ be a trajectory of the replicator dynamic when applied to a bipartite network of coordination games that has a fully mixed Nash equilibrium \mathbf{q} then the K-L divergence between \mathbf{q} and the $\mathbf{p}(t)$ is constant, i.e., does not depend on t .